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Characterising small solutions in delay differential equations through numerical approximations

Neville J. Ford & Sjoerd M. Verduyn Lunel

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Abstract

The existence of small solutions, that is solutions that decay faster than any exponential, for linear time-dependent delay differential equations with bounded coefficients depends on specific properties of the coefficients. Although small solutions do not occur in the finite dimensional approximations of the delay differential equation we show that the existence of small solutions for delay differential equations can be predicted from the behaviour of the spectrum of the finite dimensional approximations.

Keywords: Delay equations, numerical solution, small solutions

AMS subject classification: 34K28, 39A11, 65Q05

1 Introduction

Consider a linear time-dependent delay differential equation

$$x'(t) = a(t)x(t) + b(t)x(t-1), \quad t \geq s, \quad (1.1)$$

where the coefficients a and b are bounded real continuous functions. A moment of reflection indicates that the minimal amount of information of initial data in order for (1.1) to have a unique solution x is to specify a continuous function on the interval $[s-1, s]$. In fact, if $x(s+\theta) = \varphi(\theta)$, $-1 \leq \theta \leq 0$, then on the interval $[s, s+1]$ equation (1.1) reduces to an inhomogeneous ordinary differential equation

$$x'(t) = a(t)x(t) + b(t)\varphi(t-s-1), \quad s+1 \geq t \geq s. \quad (1.2)$$

This equation can be solved using the variation-of-constants formula in order to obtain

$$x(t) = e^{\int_s^t a(\sigma) d\sigma} \varphi(0) + \int_s^t e^{\int_\alpha^t a(\sigma) d\sigma} b(\alpha) \varphi(\alpha-s-1) d\alpha, \quad s+1 \geq t \geq s. \quad (1.3)$$

Once x is known on $[s, s+1]$ the same process can be repeated to find the unique solution on $[s+1, s+2]$, etc. This process is called *the method of steps* and yields a unique, globally defined, solution of (1.1). Note that, by construction, the solution becomes more smooth with increasing time.

The method of steps can be interpreted as follows. The delay differential equation (1.1) can be viewed as an iterative process on a space of continuous functions, iterating a segment of the solution. Indeed, if for $n = 1, 2, 3, \dots$, we define $x_n(\theta) = x(s+n+\theta)$, $-1 \leq \theta \leq 0$, and $a_n(\sigma) = a(n+s+\sigma)$ and $b_n(\sigma) = b(n+s+\sigma)$ for $\sigma \in \mathbb{R}$, then $x_0 = \varphi$ and for $n = 0, 1, 2, \dots$

$$x_{n+1}(\theta) = e^{\int_{-1}^\theta a_{n+1}(\sigma) d\sigma} x_n(0) + \int_{-1}^\theta e^{\int_\alpha^\theta a_{n+1}(\sigma) d\sigma} b_{n+1}(\alpha) x_n(\alpha) d\alpha, \quad (1.4)$$

$$-1 \leq \theta \leq 0.$$

From this approach, it is clear that delay differential equations are infinite dimensional dynamical systems defined by iterating a segment of a function. The method of steps also shows that, in general, solutions of (1.1) are only defined on $[s, \infty)$. Backward continuation of solutions of delay differential equations along the real axis requires smoothness of the initial function φ defined on $[-1, 0]$.

The dynamical system approach to delay differential equations is to associate with (1.1) an evolutionary system defined by the time evolution of segments of solutions. To be precise, if the solution of (1.1) with initial data $x(s + \theta) = \varphi(\theta)$, $-1 \leq \theta \leq 0$, is denoted by $x(\cdot; s, \varphi)$ and if

$$x_t(\theta; s, \varphi) = x(t + \theta; s, \varphi), \quad -1 \leq \theta \leq 0, \quad t \geq s,$$

then the solution map $T(t, s) : \mathcal{C} \rightarrow \mathcal{C}$ defined by

$$T(t, s)\varphi = x_t(s; \varphi), \quad t \geq s,$$

is an evolutionary system, i.e., $T(t, s)$ satisfies $T(s, s) = I$ and $T(t, s)T(s, \tau) = T(t, \tau)$ for $t \geq s \geq \tau$.

In this paper we are interested in the existence of a special class of solutions, so-called small solutions. A solution x of (1.1) is called a *small solution* if

$$\lim_{t \rightarrow \infty} x(t)e^{kt} = 0 \quad \text{for every } k \in \mathbb{R}. \quad (1.5)$$

The zero solution is a trivial small solution; small solutions that are not identically zero are called nontrivial. The possible existence of nontrivial small solutions for delay differential equations with bounded coefficients is important because it is a truly infinite dimensional concept. For ordinary differential equations with bounded coefficients, there cannot exist nontrivial small solutions. Furthermore, small solutions play an important role in the qualitative theory of delay differential equations. If $b(t)$ is bounded away from zero, then Cao ([1]) proved that nontrivial small solutions of (1.1) must oscillate rapidly and, in fact, must have infinitely many zeros in any unit interval. Furthermore, for autonomous equations, i.e., the coefficients a and b in (1.1) are independent of time, the existence of small solutions is closely related to the question whether the solution to (1.1) can be approximated by series of elementary solutions, that is, solutions of (1.1) of the form $x(t) = p_j(t)e^{\lambda_j t}$, where p_j is a polynomial and λ_j is a complex number. The subspace of initial functions corresponding to the elementary solutions of (1.1) is dense in the space of initial data if and only if the zero solution is the only small solution of (1.1), see Verduyn Lunel [2, 3]. Furthermore, if there exist small solutions then they have to be identically zero after finite time, see Henry [4] and Verduyn Lunel [5].

For nonautonomous equations, this last fact is no longer true, for example, Zverkin [6] showed that

$$x'(t) = \sin(2\pi t)x(t - 1), \quad t \geq 0, \quad (1.6)$$

does have nontrivial small solutions but no solutions that are identically zero after finite time.

It is our aim in this paper to understand better the possible existence of small solutions in the case that the coefficients a and b in (1.1) are periodic functions. For special classes of periodic equations there are also necessary and sufficient conditions for the existence of nontrivial small solutions, see Verduyn Lunel [7] and it is of interest to extend these conditions to larger classes of periodic equations, in particular to equations with matrix coefficients.

Since small solutions do not exist in any finite dimensional approximation of (1.1) it becomes a challenge to understand how the existence of nontrivial small solutions is reflected in the finite dimensional approximations of (1.1). In particular, is it possible to predict from the finite dimensional approximations that the limiting delay differential equation admits nontrivial small solutions? As a simple test problem to illustrate our approach we shall study the existence of nontrivial small solutions for periodic delay differential equations with the additional property that the period of the coefficients is equal to the time delay in the system. This is a first step to test our ideas and in the sequel we would like to extend our approach to systems of periodic delay equations and to periodic delay equations with more than one delay.

In Section 2 we present the analytical theory for scalar periodic delay differential equations. We discuss the Floquet theory and show that small solutions play a crucial role in the question whether there exists an autonomous equation closely related to the periodic equation. In Section 3 we discuss the numerical methods to solve delay differential equations and we explain our approach to detect from the finite dimensional approximations to the delay differential equation whether or not there exist nontrivial small solutions. In Section 4 we present the numerical evidence to justify our approach.

2 Analytical theory of periodic delay equations

In this section we consider the following class of periodic delay equations

$$x'(t) = a(t)x(t) + b(t)x(t - \tau), \quad t \geq s, \quad (2.1)$$

where the coefficients a and b are real continuous periodic functions with period τ , i.e., $a(t + \tau) = a(t)$ and $b(t + \tau) = b(t)$ for all $t \in \mathbb{R}$. We have seen that, if $x(s + \theta) = \varphi(\theta)$, $-\tau \leq \theta \leq 0$, is given, then (2.1) has a unique solution $x(t; s, \varphi)$ for $t \geq s$.

A transformation of variables allows us to reduce (2.1) even further by assuming that the coefficient a in (2.1) is identically zero. Define

$$y(t) = x(t)e^{-\int_{s-\tau}^t a(\sigma) d\sigma} \quad \text{and} \quad \widehat{b}(t) = b(t)e^{-\int_{t-\tau}^t a(\sigma) d\sigma}, \quad t \geq s - \tau, \quad (2.2)$$

then $\widehat{b}(t)$ is a real continuous periodic functions with period τ and a straightforward computation yields that y satisfies the delay differential equation

$$y'(t) = \widehat{b}(t)y(t - \tau), \quad t \geq s$$

with initial data

$$y(s + \theta) = \varphi(\theta)e^{-\int_{-\tau}^{\theta} a(s + \sigma) d\sigma}, \quad -\tau \leq \theta \leq 0.$$

Therefore, we can assume in (2.1) that $a \equiv 0$ and for further reference we state this reduced equation explicitly

$$x'(t) = b(t)x(t - \tau), \quad t \geq s. \quad (2.3)$$

Let $\mathcal{C} = C[-\tau, 0]$ denote the space of continuous functions on $[-\tau, 0]$ provided with the supremum norm. The evolutionary system associated with (2.3) is given by translation along the solution

$$T(t, s)\varphi = x_t(s, \varphi), \quad (2.4)$$

where $x_t(s, \varphi)(\theta) = x(t + \theta; s, \varphi)$ for $-\tau \leq \theta \leq 0$. Since b is a periodic function of period τ , it follows that

$$T(t + \tau, s) = T(t, s)T(s + \tau, s) \quad \text{for } t \geq s. \quad (2.5)$$

The periodicity property (2.5) allows us to define the monodromy map or period map $\Pi(s) : \mathcal{C} \rightarrow \mathcal{C}$ associated with (2.3) as follows

$$\Pi(s)\varphi = T(s + \tau, s)\varphi, \quad \varphi \in \mathcal{C}. \quad (2.6)$$

From the general theory for delay differential equations (see, for example, [8]), it follows that $\Pi(s)$ is a compact operator, i.e., $\Pi(s)$ is a bounded operator with the property that the closure of the image of the unit ball in \mathcal{C} is compact. The spectrum $\sigma(\Pi(s))$ of $\Pi(s)$ is defined to be

$$\sigma(\Pi(s)) = \{\mu \in \mathbb{C} : \mu I - \Pi(s) \text{ is not invertible}\}.$$

General facts about compact operators imply that $\sigma(\Pi(s))$ is at most countable. Furthermore, the spectrum $\sigma(\Pi(s))$ is a compact set of the complex plane with the only possible accumulation point being zero. If $\mu \neq 0$ belongs to $\sigma(\Pi(s))$, then μ is in the point spectrum of Π , i.e., there exists a $\varphi \in \mathcal{C}$, $\varphi \neq 0$, such that $\Pi(s)\varphi = \mu\varphi$.

If μ belongs to the nonzero point spectrum of $\Pi(s)$, then μ is called a *characteristic multiplier* of (2.3) and λ for which $\mu = e^{\lambda\tau}$ (unique up to multiples of $2\pi i$) is called a *characteristic exponent* of (2.3).

The generalised eigenspace $\mathcal{M}_\mu(s)$ of $\Pi(s)$ at μ is defined to be the kernel of $(\mu I - \Pi(s))^{k_\lambda}$, where k_λ is the smallest integer such that

$$\mathcal{M}_\mu(s) = \text{Ker}((\mu I - \Pi(s))^{k_\lambda}) = \text{Ker}((\mu I - \Pi(s))^{k_\lambda+1}).$$

Furthermore, the following basic properties hold

1. $\mathcal{C} = \mathcal{M}_\mu(s) \oplus Q_\mu$
2. $\dim \mathcal{M}_\mu(s) < \infty$
3. $\sigma(\Pi(s)|_{\mathcal{M}_\mu(s)}) = \{\mu\}$.

Moreover, the characteristic multipliers are independent of s and $T(t, s)|_{\mathcal{M}_\mu(s)}$ is a diffeomorphism onto $\mathcal{M}_\mu(t)$. Solutions of (2.3) with initial value in $\mathcal{M}_\mu(s)$ are of the Floquet type, namely of the form

$$x(t) = e^{\lambda t} p(t),$$

where $\mu = e^\lambda$ and $p(t) = p(t+1)$. In other words, on each generalised eigenspace $\mathcal{M}_\mu(s)$ there exists a periodic transformation that transforms the periodic equation to an autonomous equation (the Floquet theory for ordinary differential equations). In general, however, the spectrum of the period map $\Pi(s)$ does not describe the complete dynamics of the periodic delay differential equation. Since the operator $\Pi(s)$ is non-self adjoint the eigenvectors and generalised eigenvectors of $\Pi(s)$ do not necessary span the full space, or in other words, the direct sum of the generalised eigenspaces $\mathcal{M}_\mu(s)$, $\mu \in \sigma(\Pi(s))$, is not dense in $C[-\tau, 0]$. For equation (2.3) this happens if and only if there exist nontrivial small solutions if and only if the coefficient b changes sign, see [7].

To illustrate the abstract theory and to derive further information about the characteristic multipliers of (2.3), we compute the monodromy operator explicitly

$$(\Pi(s)\varphi)(\theta) = \varphi(0) + \int_{-\tau}^{\theta} b(\alpha + s)\varphi(\alpha) d\alpha \quad -\tau \leq \theta \leq 0. \quad (2.7)$$

To compute the eigenvalues of $\Pi(s)$, it suffices to solve the following equation for $\psi \neq 0$ and $z \in \mathbb{C} \setminus \{0\}$

$$z\Pi(s)\psi = \psi. \quad (2.8)$$

If we differentiate this equation with respect to θ and use representation (2.7) for $\Pi(s)$, we obtain the following nonlocal boundary value problem

$$\psi' - zb(s + \cdot)\psi = 0, \quad \psi(-\tau) - z\psi(0) = 0. \quad (2.9)$$

The solution to the differential equation in (2.9) is explicitly given by

$$\psi(\theta) = \Omega_0^\theta(z)\psi(0), \quad -\tau \leq \theta \leq 0, \quad (2.10)$$

where $\Omega_\alpha^\theta(z)$ denotes the fundamental solution to the homogeneous equation in (2.9) and is given by

$$\Omega_\alpha^\theta(z) = e^{z \int_\alpha^\theta b(s+\sigma) d\sigma}.$$

To solve the boundary condition in (2.9), we substitute representation (2.10) for ψ and conclude that we can solve the nonlocal boundary condition in (2.9) if and only if

$$z - \Omega_0^{-\tau}(z) = z - e^{-z \int_{-\tau}^0 b(\sigma) d\sigma} \neq 0. \quad (2.11)$$

Thus, the computation yields the following characterisation for the nonzero point spectrum of $\Pi(s)$

$$\sigma(\Pi(s)) \setminus \{0\} = \{\mu \mid \mu = \frac{1}{z} \text{ and } z - e^{-z \int_{-\tau}^0 b(\sigma) d\sigma} = 0\}. \quad (2.12)$$

Note that indeed, as the abstract theory predicts, the spectrum of $\Pi(s)$ is independent of s and hence, that the characteristic multipliers of (2.3) are independent of s . Also note that in example (1.6) the nonzero point spectrum of $\Pi(s)$ consists of the single point $\mu = 1$ and, hence, the eigenvectors and generalised eigenvalues of $\Pi(s)$ are not dense in the space $C[-\tau, 0]$.

To compute the characteristic exponents λ of (2.3), we substitute $\mu = e^{\lambda\tau}$ in the equation for μ given in (2.12)

$$1 = \mu e^{-\mu^{-1} \int_{-\tau}^0 b(\sigma) d\sigma}$$

and conclude that λ satisfies the following equation

$$\lambda - \widehat{b}e^{-\lambda\tau} = 0, \quad (2.13)$$

where

$$\widehat{b} = \frac{1}{\tau} \int_{-\tau}^0 b(\alpha) d\alpha. \quad (2.14)$$

It follows, in particular, that the characteristic exponents of (2.3) coincide with the spectrum of the autonomous delay differential equation

$$y'(t) = \widehat{b}y(t - \tau), \quad t \geq s, \quad (2.15)$$

where $\widehat{b} = \tau^{-1} \int_{-\tau}^0 b(\sigma) d\sigma$. However, in general, the behaviour of the Floquet solutions of, respectively, (2.3) and (2.15), is quite different, see also the discussion at the end of this section.

Indeed, substitution of an exponential solution $x(t) = e^{\lambda t}c$ into (2.15) yields that

$$[\lambda - \widehat{b}e^{-\lambda\tau}]c = 0$$

and hence λ is a zero of the characteristic equation

$$\lambda - \widehat{b}e^{-\lambda\tau} = 0. \quad (2.16)$$

Thus for each root λ of the characteristic equation the exponential $y(t) = e^{\lambda t}$ is a solution of (2.15). Since (2.16) is a transcendental equation, there are, in general, infinitely many roots.

Characteristic equations of type (2.16) are well-studied. We state here some basic results and refer to Chapter I and XI of [9] for further details.

Lemma 2.1 *The roots of the entire function (2.16) have the following properties*

- (i) *There exists a $\eta \in \mathbb{R}$, such that the right half-plane $\{z \in \mathbb{C} \mid \Re z > \eta\}$ does not contain any roots of (2.16).*
- (ii) *The number of roots of (2.16) in a given strip $\{z \in \mathbb{C} \mid \eta_- < \Re z \leq \eta_+\}$ is finite.*
- (iii) *The roots of (2.16) in the left half-plane, necessarily satisfy*

$$|\Im z| \leq Ce^{-\tau \Re z},$$

where C is a constant determined by A and B .

If we assume that the zeros of the coefficient b in (2.1) are isolated, then it follows from the necessary and sufficient condition for the existence of small solutions that equation (2.1) has nontrivial small solutions if and only if b has a sign change, see Verduyn Lunel [7].

If the coefficient b is strictly positive, then we can actually make a change of variables to transform (2.3) into the autonomous equation (2.15). Define the invertible transformation

$$\sigma(t) = (\widehat{b})^{-1} \int_s^t b(\alpha) d\alpha, \quad t \geq s - \tau, \quad (2.17)$$

and set $y(\sigma(t)) = x(t)$, then

$$\frac{dy}{d\sigma}(\sigma) = \frac{dx}{dt}(t) \frac{dt}{d\sigma} = \widehat{b}x(t - \tau)$$

and since

$$\begin{aligned} \sigma(t - \tau) &= (\widehat{b})^{-1} \int_s^{t-\tau} b(\alpha) d\alpha \\ &= (\widehat{b})^{-1} \int_s^t b(\alpha) d\alpha - (\widehat{b})^{-1} \int_{t-\tau}^t b(\alpha) d\alpha = \sigma(t) - \tau, \end{aligned}$$

it follows that $y(\sigma)$ satisfies the delay differential equation

$$y'(\sigma) = \widehat{b}y(\sigma - \tau).$$

Thus if b is positive, then (2.3) is equivalent with the autonomous delay differential equation (2.15). If b changes sign, this transformation is no longer possible and we know that the solutions of the two systems behave very differently, for example, there exist nontrivial small solutions of (2.3) that do not vanish identically after finite time, while the small solutions of the autonomous equation (2.15) are identically zero after finite time. So the nonexistence of nontrivial small solutions is a necessary condition to make a transformation of variables to an autonomous delay differential equation. This explains some of the interest in the existence of nontrivial small solutions. It is our aim to show that the existence of small solutions can be predicted from the behaviour of the spectrum of the finite dimensional approximations of (2.3).

3 Numerical methods

When one applies a standard numerical scheme with fixed step length $h = \tau/m, m \in \mathbb{N}$ to solve a delay differential equation of the form (2.3), the result is a difference equation system of finite order

$$X_{n+1} = A(n)X_n, \quad n = 0, 1, 2, \dots \quad (3.1)$$

Here the approximate values of the solution function x of (2.3) are denoted by $x_n \approx x(nh)$ and the vectors X_n are defined by

$$X_n = (x_n, x_{n-1}, \dots, x_{n-\ell})^\top.$$

The initial function values $\varphi(t), t \in [-\tau, 0]$, provide the $m + 1$ initial values

$$x_n = \varphi(nh), \quad n = -m, \dots, 0.$$

For example, the application of the forward Euler scheme, backward Euler scheme or trapezium rule (or any other linear θ -method) results in a difference scheme of order $\ell = m + 1$ and the initial function values provide the initial vector X_0

$$X_0 = (\varphi(0), \varphi(-h), \dots, \varphi(-mh))^\top.$$

In general, if we apply a k -step linear multistep method to solve (2.3) then we obtain a difference scheme of order $\ell = m + k$. In this case the initial vector X_0 (now of length $m + k$) contains the $m + 1$ initial values $x_n = \varphi(nh)$, $n = -m, \dots, 0$ augmented by values x_1, \dots, x_{k-1} obtained by some special starting procedure. To avoid such technical details in this paper, we shall confine our experiments to methods leading to difference schemes of order $m + 1$. Further details of numerical methods for delay equations may be found, for example, in [10, 11] or the many references in [12].

For an autonomous delay differential equation, the matrix $A(n)$ in (3.1) is independent of n and we can write $A(n) = A$. In this case we may seek the eigenvalues of the matrix A since these values will provide full details of the dynamical behaviour of the solutions to (3.1). One can then compare the dynamical behaviour of solutions to the delay differential equation (2.3) with that of solutions to (3.1).

For non-autonomous equations one cannot proceed in quite the same way. Here the matrix $A(n)$ varies with n . However, for periodic differential delay equations we obtain that $A(n) = A(n - m)$ for all $n > m$ and we can exploit this property and construct the matrix

$$C = \prod_{i=1}^m A(m - i).$$

It now follows that the matrix C constructed in this way is the discrete analogue of the period map (2.6) in the continuous case and we can use the eigenvalues of C to investigate the dynamical behaviour of the solution to (3.1).

Using the notation we introduced earlier in the section, we have

$$X_{n+m} = CX_n \tag{3.2}$$

and this autonomous problem is exactly equivalent (in the sense that the solution is the same whenever the initial vector is the same) to (3.1). We conclude that the dynamical behaviour of solutions to (3.1) is identical to the dynamical behaviour of solutions to (3.2).

This approach is the discrete analogue to the one that considers the behaviour of solutions to (2.3) by converting it to the form (2.15). We draw attention to the fact that, for the continuous problem, the dynamical behaviour of the periodic equation was reproduced in the autonomous equation if and only if the function b was positive (negative).

Applying a numerical method in the way we describe here is not quite so clear-cut: we have reduced the dimension of the problem from its natural infinite dimensional setting to a finite dimensional case. This means that we are sampling the function b a fixed number of times, determined by the choice of step size h . Whatever the behaviour of the function b , the numerical approximation yields a matrix C of dimension that is fixed by the choice of step size in the numerical scheme. According to the choice of step length and function b , some of the behaviour of b that is important in the continuous case may not show up in the discrete version. For example, the same sequence of sampled values $\{b_n\}$ can arise from infinitely many choices of smooth functions b that pass through the same (finitely many) sampled points.

When we analyse the autonomous difference scheme, it follows that it would be meaningless to impose a condition on the sequence of matrices $A(n)$ analogous to the condition on b that b must not change sign. Indeed, it is clear by a simple experiment, that one could construct functions \tilde{b}, \hat{b} for which \tilde{b} had no change of sign, \hat{b} had several changes of sign and $\tilde{b}(t) = \hat{b}(t)$ at all the grid points used in the numerical scheme. One must conclude that the finite dimensional numerical scheme can reproduce the *true* dynamical behaviour of the infinite dimensional continuous problem only in the limit as the step size $h \rightarrow 0$ (and the dimension consequently satisfies $n \rightarrow \infty$).

We shall consider also the *direct* discretisation of the autonomous DDE (2.15). This yields an autonomous difference scheme

$$x_{n+1} = \hat{A}x_n$$

which is more conveniently analysed here in the equivalent form

$$x_{n+m} = \hat{C}x_n, \tag{3.3}$$

where $\widehat{C} = (\widehat{A})^m$. This difference scheme will be particularly useful to us since it provides a reference set of eigenvalues of the numerical solution operator of the autonomous equation (2.15). We appeal to

Theorem 3.1 (see Theorem 3.2 of [13]) *Apply a strongly stable linear multistep method of order $p \geq 1$ to the autonomous delay differential equation*

$$y'(t) = \alpha y(t - \tau) \quad (3.4)$$

with characteristic roots that satisfy

$$\lambda - \alpha e^{-\tau\lambda} = 0. \quad (3.5)$$

For each fixed step length $h = \frac{1}{m} > 0$ the numerical method has a set S_h of $m + 1$ characteristic roots of the equation

$$\lambda^m \rho(\lambda) - h\alpha\sigma(\lambda) = 0, \quad (3.6)$$

where $\rho(\lambda)$ and $\sigma(\lambda)$ are, respectively, the first and second characteristic polynomials of the linear multistep method being used. Let λ be a root of equation (3.5) and define d_h to be the distance given by

$$d_h = \min_{s \in S_h} |e^\lambda - s^m| \quad (3.7)$$

then d_h satisfies

$$d_h = O(h^p) \quad \text{as } h \rightarrow 0. \quad (3.8)$$

The conclusion of the theorem guarantees that we can plot the $m = h^{-1}$ values from the set S_h (the eigenvalues of the matrix \widehat{C} in (3.3)) and use them as a good approximation to eigenvalues of the continuous solution operator of the equation (2.15).

Small solutions arise, as we saw in Section 2, when the eigenvalues of the period map (2.6), or equivalently, the spectrum of (2.15) does not describe the full dynamical behaviour of (2.3) and our approach to detect small solutions is to look for a difference between the eigenvalues of the matrices C in (3.2) and \widehat{C} in (3.3).

We recall that we are using a finite dimensional approximation in an attempt to identify an infinite dimensional property in the delay differential equation. Indeed, as is known, any small solution $x(t)$ satisfies $x(t)e^{\alpha t} \rightarrow 0$ as $t \rightarrow \infty$ for every real value of α and this behaviour cannot be reproduced faithfully in a finite dimensional scheme (3.1) with $A(n) = A(n - m)$ for $n > m$.

One anticipates that small solutions in the continuous problem will be approximated by small nonzero eigenvalues in the discrete scheme. As $h \rightarrow 0$ one then expects the eigenvalues corresponding to small solutions (if any) to proceed to the limit 0, while all other eigenvalues will tend to a non-zero limit equal to an eigenvalue of the continuous scheme.

We have not used variable step length schemes for simplicity of analysis. However it seems unlikely that the use of such schemes would be advantageous. In fact our discussion above indicates that any algorithm for choosing step length to give a good approximation would, in the presence of small solutions, select smaller and smaller step lengths until the lower limit on step length was attained. Therefore it is our view that the analysis of variable step length schemes for this problem reduces in any case to a consideration of schemes with fixed step length.

4 Numerical experiments

We begin this section by illustrating briefly how the known theoretical behaviour of the characteristic values of the solution map is displayed in the numerical scheme since this gives us initial insight into the behaviour that we set out to detect. Since for the simple equation that we are considering here, there is only one time delay in the system, we can rescale time and assume, in our numerical computations, that $\tau = 1$. More information about our numerical approaches is contained in our recent paper [14].

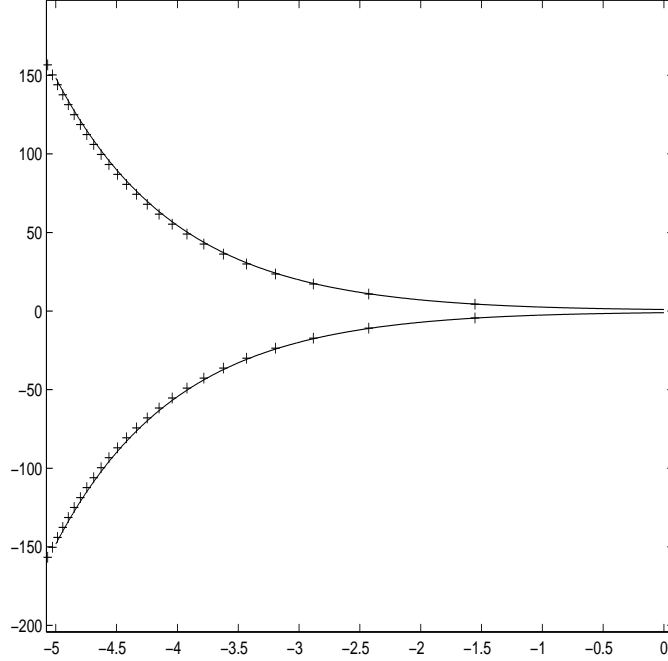


Figure 1: Locus of true characteristic values and their approximations using backward Euler scheme

For the equation

$$y'(t) = \widehat{b}y(t-1) \quad (4.1)$$

it is straightforward to show that the characteristic values lie on the locus $|\lambda| = |\widehat{b}e^{-\lambda}|$, shown (for $\widehat{b} = 1$) as a solid line in Figure 1.

According to Theorem 3.1, the corresponding characteristic values of the discrete solution (given by, for example, the backward Euler scheme) should approximate closely the true characteristic values. The approximate values (for the smallest characteristic roots when $h = \frac{1}{200}$) are marked as + in Figure 1. The values plotted here are derived by finding the product of the complex logarithm of the eigenvalues of the matrix A (c.f. Theorem 3.1) with τ/h . A second known property of the approximating characteristic values is shown (this time for $h = \frac{1}{20}$ for clarity) in Figure 2. There is one characteristic root in each horizontal band of width 2π .

The locus shown in Figure 1 is familiar from existing theory, but is not the most useful representation of the characteristic roots for our purposes. As remarked in [15], the characteristic roots approximated in this way for a fixed h -value are restricted to lie between $\pm 2\pi i\tau/h$ which is considered to be a reasonable restriction in their work for sufficiently small $h > 0$. For our purposes (and for the non-autonomous case in particular) we will plot instead values of e^λ for each characteristic value λ . This removes any ambiguity that could be caused by the incorrect choice of the branch of complex logarithm of the eigenvalues of $\prod A_n$.

For clarity we show (Figure 3) the result of replotting the loci and points derived by taking exponentials of the values shown in Figure 1. Figure 3 is not so clear close to the origin and so we show a zoomed version of the points as Figure 4.

Now we can use these figures as a basis for investigating a non-autonomous version of the equation that possesses small solutions. Suppose b satisfies:

$$b(t) = \begin{cases} 2 + \epsilon & 0 \leq t < \frac{1}{2} \\ -\epsilon & \frac{1}{2} \leq t < 1 \end{cases} \quad (4.2)$$

then $\int_0^1 b(s)ds = 1$ and so the corresponding autonomous problem is the one we considered earlier in this section. However, for any $\epsilon > 0$ it will turn out that the non-autonomous scheme has small solutions and we seek to exhibit this property in the discrete scheme.

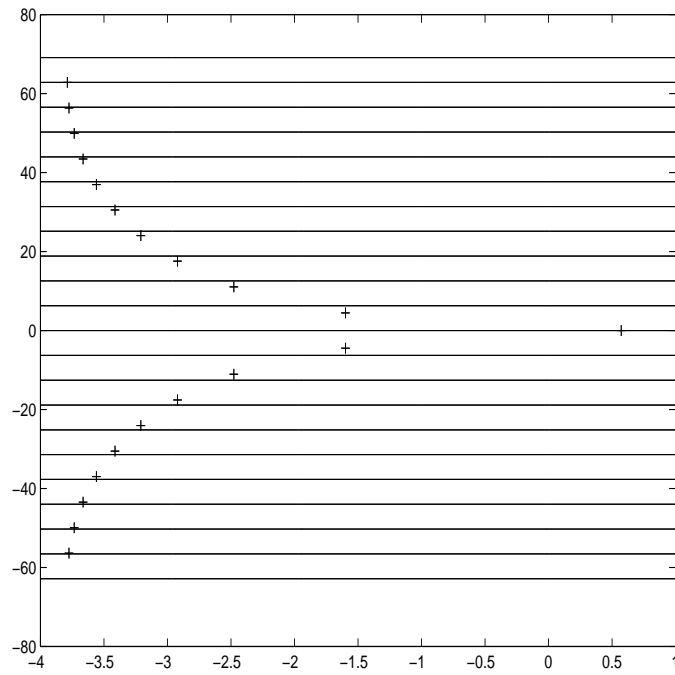


Figure 2: Approximations to characteristic values using backward Euler scheme

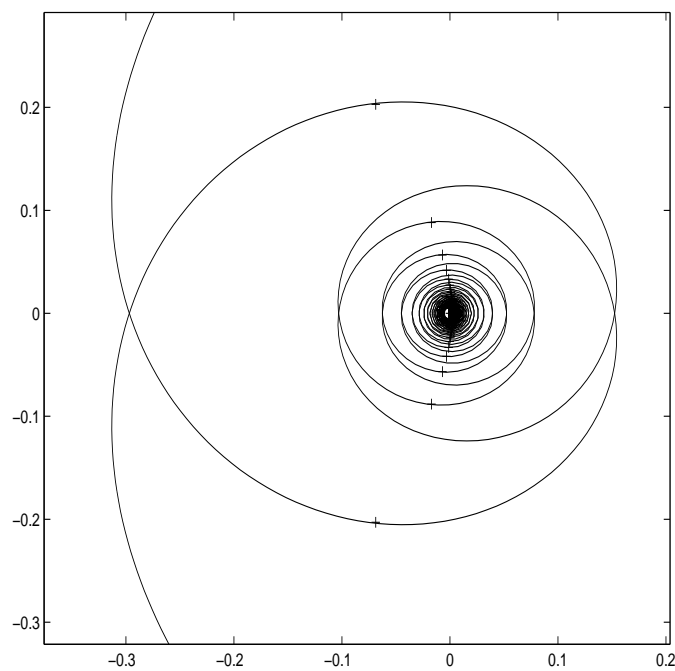


Figure 3: Locus of exponentials of true characteristic values and their approximations using backward Euler scheme

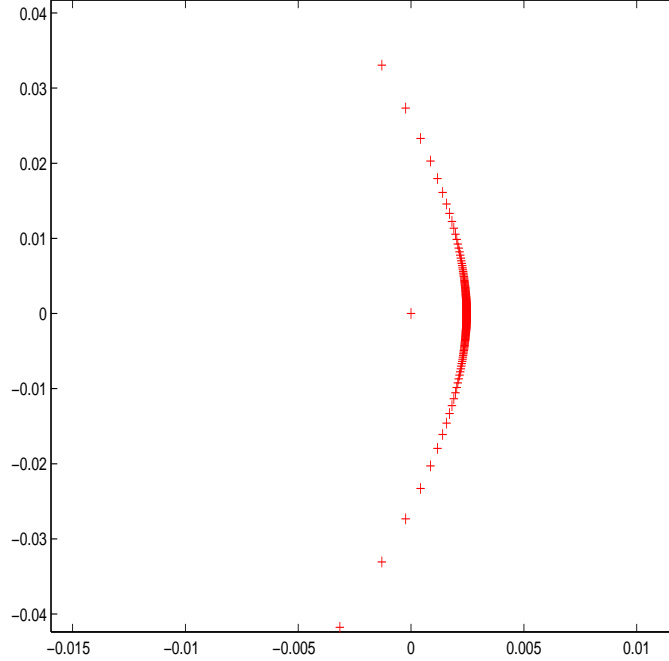


Figure 4: Zoomed approximations of exponentials of characteristic values using backward Euler scheme

Figure 5 shows the outcome when $\epsilon = 1$. Notice that there are three distinct trajectories. The central locus from the autonomous scheme (marked with \times) is approximated by the right-hand locus from the non-autonomous scheme. Indeed, for smaller step-sizes, the approximation becomes closer to the extent that the two loci nearly coincide. The left hand locus from the non-autonomous scheme does not approximate any corresponding locus from the autonomous scheme and we propose this as a method for identifying the existence of small solutions.

One can now vary the value of ϵ . We illustrate with $\epsilon = 0.5, 0.2, 0$

The results shown in the plots support our conjecture that the existence of small solutions corresponds to the observation of an extra locus for the non-autonomous equation. Note how the extra locus is not present for the case $\epsilon = 0$ where no small solutions are present.

In conclusion, we would like to remark that experiments using alternative numerical methods give the same results. We have considered the use of the trapezium rule and other θ -methods for the same form of function b . We have also considered (see for example [14]) the use of smooth periodic functions b , always with similar results as presented here.

5 Concluding remarks

We have seen that the dynamics of a periodic delay differential equation can sometimes be described by an autonomous delay differential equation. In fact for the simple equation (2.3) the dynamics can be described by an autonomous delay differential equation if and only if the coefficient b does not change sign if and only if there are no nontrivial small solutions. Although coefficients that change sign do not have such a strong effect on the dynamics of the finite dimensional approximations of the delay differential equation using numerical schemes, we show that studying the eigenvalue distribution of the simple numerical schemes which approximate the periodic delay differential equation is quite effective in helping to identify whether the dynamics of the periodic delay equation is described by an autonomous delay differential equation. We hope that our approach studying the eigenvalue distribution and maybe resolvent estimates for the finite dimensional approximations can be used to identify for more general classes of periodic equations whether or not the dynamics is described by

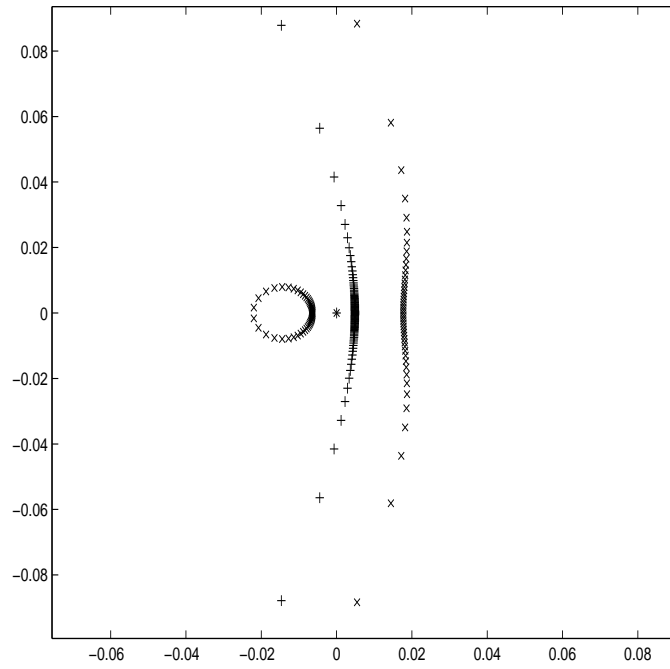


Figure 5: Approximate characteristic values, marked $+$ for autonomous scheme and \times for non-autonomous scheme

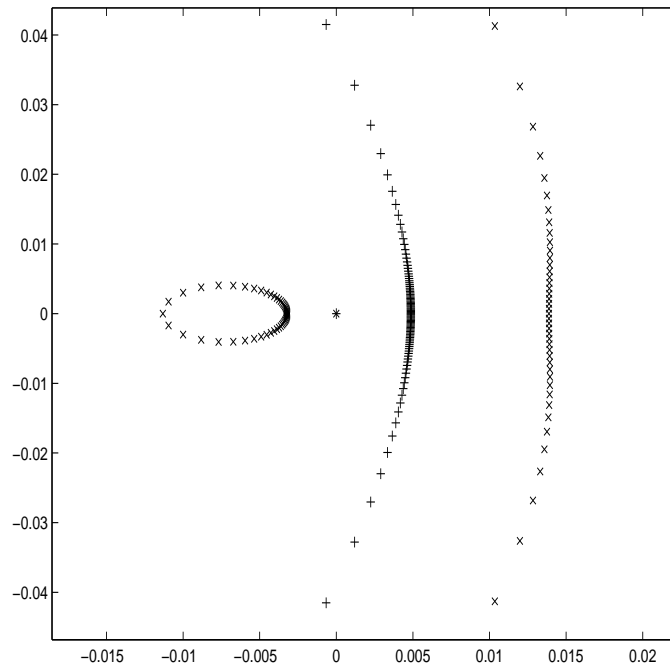


Figure 6: Same as Figure 5 but with $\epsilon = 0.5$

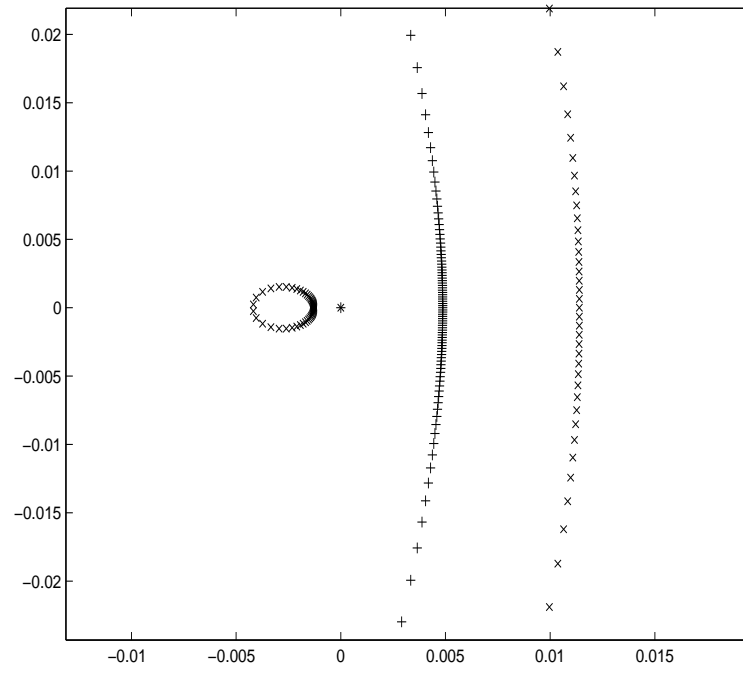


Figure 7: Same as Figure 5 but with $\epsilon = 0.2$

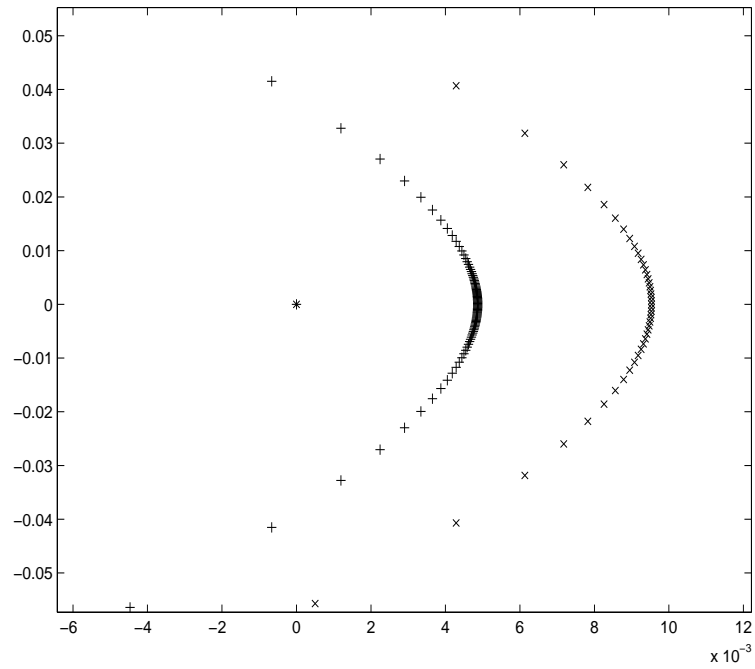


Figure 8: Same as Figure 5 but with $\epsilon = 0$

an autonomous delay differential equation.

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